Review

of a collection of selected papers

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1 Diplomas and scientific degrees

1. MSc in Technical Physics at the Faculty of Technical Physics and Applied Mathematics, Technical University of Lodz, Poland in 1992. The advisor of my master thesis entitled 'Quantum mechanics in general theory of relativity. Path integrals' was dr. hab. M. Przanowski.

2. PhD in Physics at the Faculty of Physics and Chemistry at the University of Lodz in 1998. I presented the thesis 'The generalised Weyl– Wigner– Moyal formalism in curved phase spaces' My advisor was dr. hab. M. Przanowski.

2 Career

1. Junior assistant at the Institute of Physics, Technical University of Lodz, 1992.

2. Assistant at the Institute of Physics, Technical University of Lodz, 1992 – 1998.

3. Lecturer assistant at the Institute of Physics, Technical University of Lodz, from 1998.

3 Sequence of selected papers

Applications of the Fedosov construction of deformation quantization in quantum mechanics

List of selected publications

L1. J. Tosiek, *The Weyl – Wigner – Moyal formalism: quantum mechanics on curved phase spaces*, Publ. Real Soc. Mat. Esp. **3** (2001), 195–211.

L2. M. Gadella, M.A. del Olmo, J. Tosiek, *Quantization on a 2 dimensional phase space with a constant curvature tensor*, Ann. Phys. **307** (2003), 272–307.

L3. J. Tosiek, *The Weyl bundle as a differentiable manifold*, J. Phys. A: Math. Gen. **38** (2005), 5193–5216.

L4. J. Tosiek, *Abelian connection in Fedosov deformation quantization. I. The 2-dimensional phase space*, Acta Phys. Pol. B **38** (2007), 3069–3086.

L5. J. Tosiek, *The Fedosov $*$-product in Mathematica*, Comp. Phys. Comm. **179** (2008), 924–930.

L6. J. Tosiek, *The Fedosov $*$-product in Mathematica*, Comp. Phys. Comm. **181** (2010), 704.

L7. J. Tosiek, *Compatible symplectic connections on a cotangent bundle and the Fedosov quantization*, J. Math. Phys. **52** (2011), art no. 022107.

3.1 Introduction

The first quantization procedure called the **canonical quantization** was proposed by Weyl [1], Dirac [2] and von Neumann [3]. Unfortunately, the idea of assigning self − adjoint operators acting in the Hilbert space $L^2(\mathbb{R}^n)$ to classical observables was based on incompatible rules [4].

Hence the canonical quantization program was substituted by operator ordering rules [5], [6]. However, these ordering rules are based on the Fourier transform. Thus they cannot be generalized for systems in phase spaces different from $\mathbb{R}^{2n}$.

To construct quantum mechanics of systems in arbitrary phase spaces we have to choose one of two options: geometric quantization[1] or deformation quantization. My scientific activity is focused on the latter option.

Deformation quantization for systems in the phase spaces $\mathbb{R}^{2n}$ was formulated in the first half of the previous century. E. P. Wigner [9] proposed a quantum state description with the use of a quasi-probability function. Some years later Groenewold [10] and Moyal [11] introduced a $*$ − product nowadays called the **Moyal product**.

Next years several interesting papers devoted to deformation quantization had been published, but a new era began when Bayen *et al* presented their outstanding articles [12], [13]. In these works quantum mechanics appeared a deformation of classical physics. Thereafter many scientists, especially mathematicians, started intensive investigation of this new area (see e.g. review [14]). M. de Wilde and P. Lacomte [15] proved the existence of a formal deformation of a Poisson algebra in an arbitrary symplectic manifold, B. Fedosov [16], [17] proposed a recurrent method of calculating the $*$ − product in an arbitrary symplectic manifold and M. Kontsevich [18] showed the existence of a nontrivial deformation of the usual multiplication of functions in any Poisson manifold.

Mathematicians are mainly focused on questions about existence and equivalence of $*$ − products. In my research I concentrate on applications of deformation quantization, especially of the Fedosov construction, in quantum physics. In the articles included in this report I analysed methods of choice of a symplectic connection in the phase space. Moreover, I proposed a matrix version Fedosov's algorithm. I also prepared a computer program to calculate symbolically the $*$ − product and I applied deformation quantization method to solve two quantum examples: the issue of eigenstates of a 1 − D harmonic oscillator and the question of eigenstates of momentum and position in a curved symplectic space.

Numbers of formulas in the report start with the letter W as distinct from formulas taken from the papers from the list. References containing the letter 'L' come from the list of selected papers.

Notation in this report is in agreement with the notation used in the papers. Hence e.g. when I wrote about compatible symplectic connection coefficients from the article L7, I put γ_{ijk}. But in the example from the paper L2 I applied the symbol Γ_{ijk}. Moreover, in the publications L1–L4 instead of the Fedosov convention saying that components of tangent vectors are denoted by y^i, I used X^i.

Paper L1 presenting my talk given at the IX Fall Workshop on Geometry and Physics in Vilanova i la Geltrú 2000, was based on my PhD thesis. But a new result − the solution of the 1 − D eigenvalue equation of the Hamilton function of a harmonic oscillator in curvilinear coordinates in frames of the Fedosov construction was also presented. In my report I refer only to this example.

3.2 Preferred symplectic connections in a phase space

A symplectic connection in the phase space is the initial data for the Fedosov construction. Contrary to the Riemannian geometry, an arbitrary symplectic manifold can be equipped with many symplectic connections [19]. A natural question arises, which symplectic connection is the most suitable for the construction of a $*$ − product. In my papers I considered two criteria.

As it is known, the Fedosov algorithm is based on a recurrence. Hence the most desirable symplectic connection is a connection generating a final Abelian connection series. It means that the iterative

[1] A complete course of deformation quantization can be found in the books by N. Woodhouse [7] and J. Śniatycki [8].

procedure determining the correction r to the symplectic connection at a step stops producing new elements. Notice that the term 'finite' refers not only to a maximal power of the deformation parameter $\hbar$ in the Abelian connection series but refers to a finite degree of series in a sense of the definition of the degree of a series by Fedosov [16], [17].

In my paper L4 I considered a question of existence of such a symplectic connection. I proposed the system of equations (3.8a) – (3.8d), which is a necessary and sufficient condition for the existence of a finite Abelian connection.

Corollaries 3.4 and 3.5 following from this system of equations are necessary conditions for the existence of an infinite Abelian connection series. Moreover, I presented probably the only one known example of a finite Abelian connection series. Such a series can be generated in a symplectic manifold $\mathcal{M}$ of a dimension at least 4. Locally this Abelian series is determined by a symplectic connection Γ, which only coefficients

$$\Gamma_{l_1 l_2 l_3}(q^{l_4}, \ldots, q^{l_s}), \quad 1 \leq l_1, \ldots, l_s \leq \dim \mathcal{M},$$

are different from 0. For two arbitrary indices l_i, l_j the Poisson bracket of coordinates $\{q^{l_i}, q^{l_j}\}$ disappears.

Let the symplectic curvature tensor of the connection Γ be denoted by R_Γ. If for a number $4 \leq z$ the condition

$$(\partial_\Gamma \delta^{-1})^{z-3} R_\Gamma = 0,$$

is satisfied, then the Abelian connection series based on the symplectic connection Γ is finite.

In article L4 I also considered the case of a 2– D symplectic manifold. Applying Proposition 4.1 I proved the main theorem of the paper saying that in a 2– dimensional curved symplectic manifold any Abelian connection calculated by Fedosov's algorithm is an infinite formal series.

Although this result has been achieved for a 2– D curved symplectic manifold, in many cases it helps to predict an infinite form of the Abelian connection in more dimensional phase spaces.

It is worth mentioning formula (4.4) being an explicit form of the Moyal product of polynomials $(x^i)^j (p_i)^k * (x^i)^l (p_i)^r$. I applied this result in papers L5–L7.

Simplicity is a subjective criterion and it should not be the only one condition imposed on a choice of a symplectic connection. For example in Refs. [20], [21] have been presented symplectic connections compatible with additional structures in a symplectic space and symplectic connections following from a variational principle.

In my considerations devoted to a physically reasonable choice of a symplectic connection I focused on systems, which phase spaces are cotangent bundles. The base spaces for such systems are configuration spaces and these spaces are usually equipped with a metric or at least a symmetric connection.

In paper L7 I presented a symplectic connection 'containing' the linear symmetric connection from the base space. At the beginning I had to construct a proper Darboux atlas defined in Def. 3. Such an atlas can be introduced in any cotangent bundle. In Darboux charts spatial and momenta coordinates are distinguished.

In each Darboux chart I proposed the symplectic connection locally determined by the connection coefficients

$$\gamma_{IJK} = 0 \ , \ \gamma_{IJ\alpha} = 0 \ , \ \gamma_{I\alpha\beta} = -\Gamma^{I-n}_{\alpha\beta}(q^1, \ldots, q^n) \ , \ \gamma_{\alpha\beta\delta} = p_\epsilon f^\epsilon_{\alpha\beta\delta}(q^1, \ldots, q^n). \tag{W.1}$$

By Latin capital letters I denote momenta indices, small Greek indices are spatial indices and the dimension of the configuration space equals n. Symbols $\Gamma^{I-n}_{\alpha\beta}$ represent the coefficients of a linear symmetric connection in the base space. By $q^1, \ldots, q^n$ I mean coordinates in the configuration space. Quantities $f^\epsilon_{\alpha\beta\delta}$ are geometric objects. The Einstein summation convention in (W.1) has been applied.

Point transformations (11) between proper Darboux charts do not change a form of coefficients (W.1). Only the symplectic connection coefficients $\gamma_{\alpha\beta\delta}$ can be chosen arbitrarily. The construction of the symplectic connection, although done locally, can be extended in the whole symplectic manifold.

The symplectic connection (W.1) is called a **compatible symplectic connection**. The adjective 'compatible' refers to the linear symmetric connection in the base manifold.

In Sec. 2 of paper L7 I proved several properties of the compatible symplectic connection. These properties are presented in Propositions 1–3. I showed that for an arbitrary vector field parallel transported along the curve c in the cotangent bundle $\mathcal{T}^*\mathcal{M}$, the canonical projection of this field on the base space $\mathcal{M}$ is parallel propagated along the projection of the curve c. Moreover, I proved that every compatible symplectic connection is homogeneous (see Def. 4) and that relations (23) are satisfied. A natural resume of the mentioned facts is Theorem 1, which can be treated as a geometric definition of the compatible symplectic connection.

In the next step I analysed properties of a symplectic curvature tensor $K_{ijkl},\ \ 1 \leq i,j,k,l \leq 2n$. As I showed, this tensor is determined by its coefficients of the kind $K_{\alpha\beta\delta I}$ and the linear connection from the base space.

It is known [16], [17], that the Abelian correction r to the symplectic connection appears only if the symplectic space is curved. Hence I considered the question, if having a curved configuration space we are able to define a flat compatible symplectic connection. I proved that the symplectic connection may be flat only if the base space is flat. Moreover, as I wrote in Theorem 2, for a given flat linear symmetric connection in the configuration space there exists only one flat compatible symplectic connection. In this special case the compatible symplectic connection coefficients $\gamma_{\alpha\beta\delta}$ are determined by the symplectic connection coefficients $\gamma_{\epsilon+n\,\mu\nu}$ and their partial derivatives.

Finally, I found a sufficient and necessary condition

$$\forall_{\alpha,\beta} \sum_{\epsilon=1}^{n} K_{\alpha\beta\epsilon\epsilon+n} = 0$$

for the compatible symplectic connection to be Ricci flat.

Paper L7 contains two examples of a compatible symplectic connection. The first one is the symplectic connection induced by a linear connection. This induced symplectic connection is the only one flat compatible symplectic connection for a given flat symmetric connection. The induced symplectic connection was proposed by Yano and Ishihara [22]. In deformation quantization this kind of symplectic connection has been analysed in Refs. [23] and [24].

The second example refers to a narrow class of cotangent bundles, for which their base spaces can be covered by an atlas consisting of charts with linear transition functions between them. Thus the compatible symplectic connection coefficients can be chosen as in formula (43). In this special case not only the connection coefficients γ_{IJK} and $\gamma_{IJ\alpha}$ disappear but so do elements $\gamma_{\alpha\beta\delta}$.

In the next part of the paper I investigated properties of an Abelian connection generated by a compatible symplectic connection according to the Fedosov algorithm. I found that only three kinds of the symplectic curvature 2–form terms appear. They are presented on page 16. Moreover, there are also three types of components of the Abelian correction r 1–form (see p. 18). Thus for a compatible symplectic connection the recurrent algorithm reduces to the loop presented on page 20. As it can be seen, in this case the Abelian connection does not contain the deformation parameter $\hbar$.

It is an interesting question, inspired by the previously presented article L4, whether there exist compatible symplectic connections generating finite Abelian connection series. For a 3–D configuration space and the induced symplectic connection in its cotangent bundle such a finite Abelian connection series is generated by the symmetric connection locally determined by the coefficients

$$\Gamma^3_{11}(q^1,q^2)\ \ ,\ \ \Gamma^3_{22}(q^1,q^2)\ \ ,\ \ \Gamma^3_{12}(q^1,q^2). \tag{W.2}$$

Functions $\Gamma^3_{11}, \Gamma^3_{22}, \Gamma^3_{12}$ are polynomials of variables q^1, q^2. By q^1, q^2, q^3 are denoted local coordinates in the configuration space. I published this example in Ref. [25].

Sec. 5 of paper L7 contains several conclusions about a form of the $*$–product determined by a compatible symplectic connection. I emphasize the observation, that the maximal power of the deformation parameter $\hbar$ appearing in the $*$–product is determined by the maximal powers of momenta. For the

$*$–product

$$\Big[(p_1)^{i_1}\dots(p_n)^{i_n}f(q^1,\dots,q^n)\Big]*\Big[(p_1)^{j_1}\dots(p_n)^{j_n}g(q^1,\dots,q^n)\Big]$$

the highest power of $\hbar$ equals $\hbar^{i_1+\dots+j_n}$. Hence the $*$–product of functions depending exclusively on spatial coordinates reduces to the usual commutative product of functions. Moreover, the canonical variables $q^1,\dots,p_n$ satisfy Dirac's commutation rules

$$\{q^\alpha,q^\beta\}_M=0\ ,\ \{q^\alpha,p_\beta\}_M=-i\hbar\delta^\alpha_\beta\ ,\ \{p_\alpha,p_\beta\}_M=0, \tag{W.3}$$

where the symbol $\{a,b\}_M:=a*b-b*a$ denotes the Moyal bracket[2].

Yet a deformation is seen in the $*$–product of momenta. In general $p_\alpha*p_\beta=p_\alpha\cdot p_\beta+\hbar^2 f$, where the function f depends on linear symmetric connection coefficients in the base space $\mathcal{M}$.

I also analysed a general form of Abelian connections generated by special kinds of compatible symplectic connections described in the 3rd Section. For an induced symplectic connection the series r does not contain terms such as $r[0|,i_1,\dots,i_n|0|\alpha+n]$.

For a compatible symplectic connection, which does not have components of the type $\gamma_{\alpha\beta\delta}$, functions standing at differential operators in $B_i(\cdot,\cdot)$'s in the $*$–product depend only on spatial coordinates.

3.3 The Weyl bundle as a differentiable manifold

Application of the original Fedosov algorithm requires a lot of effort and patience. Hence I prepared a matrix version of Fedosov's idea. Some preliminary considerations on this topic were contained in paper [26], but the complete construction I published in Ref. L3.

Fedosov's scheme is based on an algebraic method. One uses the $\circ$– product defined in the Weyl algebra bundle to calculate a series representing an Abelian connection. This operation is based on an iterative formula. Then, applying another recurrent rule one obtains flat sections of the Weyl bundle. Flat sections $\sigma^{-1}(f_1),\sigma^{-1}(f_2)$ represent formal series f_1 and f_2 defined in the symplectic manifold. Finally we $\circ$– multiply these flat sections $\sigma^{-1}(f_1)\circ\sigma^{-1}(f_2)$ and project their product on the center of the Weyl algebra. The projection is the $*$– product f_1*f_2 of the formal series.

The algebraic way is not the only possible one. As I showed in my article L3, the Fedosov construction can be formulated in frames of differential geometry. The key point is defining a differentiable structure in the Weyl bundle. The main difficulty is caused by the infinite dimension of the Weyl bundle as a differentiable manifold. Infinite dimensional manifolds are known in physics. An example is the separable Hilbert space of series l^2. Yet this Hilbert space is modelled on a Banach space and the Weyl bundle is not normalizable.

At the beginning I introduced a differential structure in the space $\mathbb{R}^\infty$. This vector space is a natural generalisation of spaces $\mathbb{R}^n,\ \ n\in\mathcal{N}$. To achieve my goal I constructed a topology in $\mathbb{R}^\infty$. I did it in two equivalent ways: with the use of a basis of neighbourhoods (following K. Maurin [27]) and with a series of seminorms [28]. Therefore the space $\mathbb{R}^\infty$ is a topological space with the Tichonov topology.

In the next step I proved that the space $\mathbb{R}^\infty$ with the Tichonov topology is a Hausdorff space. It implies that $\mathbb{R}^\infty$ is a Fréchet space. Then I proposed an atlas in the space $\mathbb{R}^\infty$ (see page 5197). The transition functions are linear bijections. Due to the fact that the transition functions are linear, I was able to introduce partial derivatives $\frac{\partial x^j}{\partial y^i}$. Thus the space $\mathbb{R}^\infty$ is indeed a C^∞ differentiable manifold.

In Sec. 3 I equipped the Weyl bundle with a differentiable structure. First I observed, that elements of the Weyl algebra can be treated as symmetric tensors from spaces $(T^*_{\mathtt{p}}M)^l:=\underbrace{T^*_{\mathtt{p}}M\odot\cdots\odot T^*_{\mathtt{p}}M}_{l-\text{times}}$, where $T^*_{\mathtt{p}}M$ denotes the cotangent space of a manifold M at the point $\mathtt{p}$. Each space $(T^*_{\mathtt{p}}M)^l,\ l\geq 0$ is a metric space with the distance (3.12). Thus every space $(T^*_{\mathtt{p}}M)^l,\ l\geq 0$ is a topological space. Moreover, it can be equipped with the norm given at page 5198 so it becomes a complete space.

[2]The Moyal bracket is often defined as $\{a,b\}_M:=\frac{1}{i\hbar}(a*b-b*a)$. Then $\{q^\alpha,p_\beta\}_M=-\delta^\alpha_\beta$. The sign '–' comes from the convention used by Fedosov.

After similar considerations I concluded that the Weyl space

$$P^*_{\mathrm{p}}M[[\hbar]] := \bigoplus_{i=0}^{\infty}\Big(\bigoplus_{l=0}^{\infty}(T^*_{\mathrm{p}}M)^l \oplus (T^*_{\mathrm{p}}M)^l\Big)$$

is a Fréchet space with the metric (3.20). Moreover, this space can be covered with one chart (3.21). Rules of a choice of coordinates are given on page 5200.

Finally I proved that the $\circ-$ product is continuous in the Weyl algebra.

The union

$$\mathcal{P}^*\mathcal{M}[[\hbar]] := \bigcup_{\mathrm{p}\in\mathcal{M}} P^*_{\mathrm{p}}M[[\hbar]]$$

called the **Weyl bundle** is a differentiable manifold. It is also a vector bundle. Its structure has been analysed in Sec. 4.

Next I proposed a construction of a symplectic connection matrix in the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$. The base space of the Weyl bundle is a Fedosov manifold i.e. it is a symplectic manifold equipped with a symplectic connection.

As I showed in formula (5.71), the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is a double direct sum of vector bundles

$$\mathcal{T}^*\mathcal{M}^l := \bigcup_{\mathrm{p}\in\mathcal{M}} (T^*_{\mathrm{p}}M)^l.$$

Hence it can be equipped with the induced symplectic connection represented by the matrix (5.72).

Therefore, applying the Fedosov recurrent formula we obtain the Abelian connection matrix (6.80). With the use of this matrix we can find flat sections of the Weyl bundle, which represent functions f_1 and f_2 respectively and finally, from (6.81), we obtain the product $f_1 * f_2$.

My success in the context of paper L3 is reformulation of Fedosov's construction in terms of differential geometry. Calculation of the $*$–product of observables has been reduced to operations on infinite dimensional matrices. On the other hand, this geometric reformulation did not simplify calculations. The reason is the complicated definition of the $\circ-$ product (see formula (3.23)).

3.4 The computer program calculating the $*-$ product according to the Fedosov method

A recurrent character of the Fedosov algorithm predisposes it to be written in a form of a computer program. The first attempt to preparing a program finding the $*-$ product according to the Fedosov model I made when I was preparing papers L1 and L2. However, this program was not universal enough so I decided not to publish it. Some years later I turned back to the idea of preparing such a program. I chose the Mathematica by Wolfram as its compiler.

I assumed that the program would perform symbolic calculations in a symplectic space of a dimension given by a user to a power of $\hbar$ proposed by the user. At the beginning I considered the possibility of doing computations in an arbitrary chart. However, due to limitations of the efficiency of a computer I restricted to the physically important case of Darboux charts. The first version of my program was described and published in Ref. L5. One and a half years later I presented the second, optimized version of the program. The latter version was announced in paper L6. Every half a year I receive a report from the journal Computer Physics Communications saying, how many times this program was imported from the library. On average it is downloaded once for two months.

There were two main difficulties I had to deal with: preparation of a universal module calculating the $\circ-$ product for an arbitrary dimension of the phase space and optimizing of calculations. Due to an enormous number of operations I divided multiplied terms in classes containing the same elements $(y^1)^{i_1}\dots(y^n)^{i_n}$. Moreover, I applied the observation presented in Ref. L4, that the $\circ-$ product of symmetric tensors can be substituted by the $\circ-$ product of polynomials. I also paid attention that the program did exclusively operations necessary for finding the $*-$ product up to a fixed power of $\hbar$.

The user can see not only the final $*-$ product, but he/she may follow computations. Therefore the symplectic connection 1– form, the symplectic curvature 2– form and the flat sections $\sigma^{-1}(a), \sigma^{-1}(b)$ representing multiplied functions are shown on a screen. Paper L5 contains a test of efficiency of the program.

3.5 Physical applications of the Fedosov method

The Fedosov method has been proposed as an algorithm enabling calculating the $*-$ product in a curved symplectic manifold. However, it can be applied also in flat symplectic spaces. Its main advantage is working in an arbitrary chart. Hence it is used e.g. for calculating the Moyal product directly in curvilinear coordinates in the phase space $\mathbb{R}^{2n}$. An interesting example of an application of the Fedosov method to a flat problem is finding eigenvalues and eigenfunctions of the Hamilton function of a 1–D harmonic oscillator. I presented this example in paper L1.

The $*-$ eigenvalue equation of the Hamilton function

$$H(q,p) = \frac{p^2}{2m} + \frac{mq^2}{2}$$

is of the form

$$\left(\frac{mq^2}{2} + \frac{p^2}{2m}\right) W_E(q,p) + \frac{i\hbar}{2}\left(\frac{p}{m}\frac{\partial W_E(q,p)}{\partial q} - mq\frac{\partial W_E(q,p)}{\partial p}\right) + \\ -\frac{\hbar^2}{8}\left(m\frac{\partial^2 W_E(q,p)}{\partial p^2} + \frac{1}{m}\frac{\partial^2 W_E(q,p)}{\partial q^2}\right) = E \cdot W_E(q,p) \tag{W.4}$$

with the auxiliary condition

$$\{H, W_E(q,p)\}_M = 0. \tag{W.5}$$

By $W_E(q,p)$ I mean the Wigner eigenfunction of hamiltonian corresponding to the eigenvalue E. Eqs. (W.4) and (W.5) are partial differential equations.

I assumed that the phase space $\mathbb{R}^2$ of the harmonic oscillator is symplectic flat and that in the coordinates (q,p) all of symplectic connection coefficients disappear. Under these assumptions in the coordinates (q,p) from the Fedosov scheme we get formula (W.4).

A new chart (H, φ) can be obtained from the chart (q,p) with the use of the transformation rules

$$\begin{cases} q &= \sqrt{\frac{2H}{m}}\sin\phi \\ p &= \sqrt{2mH}\cos\phi. \end{cases}$$

Applying the transformation rule (6.56) for the symplectic connection I found that the symplectic connection 1– form in the new coordinates (H, ϕ) is represented by the formula

$$\Gamma = \frac{1}{4H}X^2X^2 d\phi + \frac{1}{2H}X^1X^2 dH + HX^1X^1 d\phi.$$

In article L1 I applied the Fedosov method to derive equation (W.4) directly in coordinates (H, φ). This equation is of the form (6.61) and the auxiliary condition (W.5) is in this case equivalent to the requirement that the Wigner function is real (see formula (6.62)).

As it can be seen, the change of coordinates $(q,p) \to (H, \varphi)$ plus condition (W.5) turned the eigenvalue equation (W.4) into the ordinary differential equation

$$(H - E)W_E(H) - \frac{\hbar^2}{4}\frac{\partial W_E(H)}{\partial H} - \frac{\hbar^2}{4}H\frac{\partial^2 W_E(H)}{\partial H^2} = 0.$$

Its solutions are well known.

The change of coordinates from the chart (q,p) to the new ones (T, H), where H is a Hamilton function and T is an arrival time, seems to be a more general procedure, which helps solving the eigenvalue equation of a 1–D Hamiltonian $H = \frac{p^2}{2m} + V(q)$. Some new results I present in the paper [29] (in preparation).

A task, I was faced with next, was a choice of physically reasonable solutions of Eqs. (W.4) and (W.5). The condition of reality imposed on a Wigner function combined with relation (6.64) and the condition of normalization (6.69) for a discrete spectrum or the condition of reality plus formula (6.65) are not useful, because every Wigner function contains negative powers of $\frac{1}{\hbar}$ of an arbitrary index.

To deal with the problem of choice of solutions I used the fact that the Moyal product is closed. Hence Wigner functions of pure states satisfy one of the integral conditions (6.68). As I showed, functions (6.70) fulfilling Eq. (6.63) and assigned to the eigenvalues (6.71) satisfy the integral conditions (6.68) for a discrete spectrum. Moreover, they constitute a complete system of functions. Thus there are not any other eigenstates of the harmonic oscillator.

My result is in agreement with the well known solution obtained in frames of the Hilbert space formulation of quantum mechanics. I achieved it applying only methods of deformation quantization. The example of the 1–D harmonic oscillator confirms the statement, that deformation quantization is an autonomous mathematical model of quantum physics.

Paper L2 has been devoted to a problem of calculating eigenvalues and Wigner eigenfunctions of position and momentum in a 2–D curved phase space $\mathbb{R}^2$. Although in each 2–D symplectic space every $*$–product is equivalent to the Moyal product [30], the equivalence of $*$– products does not imply equality of eigenvalues nor Wigner eigenfunctions [31]. Hence the quest for the eigenvalues and the Wigner eigenfunctions of position and momentum in a 2–D curved phase space $\mathbb{R}^2$ is justified.

I started from a choice of a symplectic connection. Since the phase space was a cotangent bundle, I decided to equip it with a compatible symplectic connection. When I was working in this problem, I did not have a complete idea of a compatible symplectic connection, but I knew what should have been expected from a physically reasonable symplectic connection. The configuration space was the $\mathbb{R}$ axis, so I put the symplectic connection coefficient $\Gamma_{211} = 0$. From the reasons explained in paper L7 I also fixed up the coefficients $\Gamma_{122} = \Gamma_{222} = 0$. More complicated was a nontrivial choice of the function Γ_{111}. I decided to consider the case $\Gamma_{111} = p$, which resulted in a constant curvature tensor. This choice enabled me also an easy analysis of the flat limits: $\lim_{a\to 0^-} ap$ and $\lim_{a\to 0^+} ap$.

As the atlas consisted of one chart, I need not apply the partition of unity to introduce the component Γ_{111} globally. The compatible symplectic connection determined by the proposed coefficients is curved. The only non vanishing component of the symplectic curvature tensor is

$$R_{1112} = -\frac{1}{4}.$$

The only nonzero Ricci tensor component is $K_{11} = \frac{1}{4}$.

An explicit form of the Abelian connection is the series (4.17). A non recurrent formula determining coefficients of the Abelian connection is related to the Catalan numbers and it is of the form (4.16). Nowadays I see that the series (4.17) confirms the result presented in paper L7 according to the from of the Abelian connection generated by a compatible symplectic connection.

In the next step I found the series $\sigma^{-1}(p)$ representing momentum and I proved that, due to the relationship

$$i\hbar\frac{\partial W_{\mathbf{p}}(q,p)}{\partial q} = 0,$$

the eigenvalue equation of momentum is of the form (4.27). By $W_{\mathbf{p}}(q,p)$ I denoted the Wigner eigenfunction associated to the eigenvalue $\mathbf{p}$.

Equation (4.27) is an ordinary differential equation of an infinite degree and I had no idea, how to solve it. Fortunately, the eigenvalues and the Wigner eigenfunctions of momentum p can be found from the eigenvalue equation of p^2. Indeed, $p * p = p \cdot p$ so momentum and its square commute. The eigenvalue equation of p^2 is the modified Bessel equation (4.29). Its general solution is the linear combination (4.30) of a modified Bessel function with the imaginary parameter $\frac{2i\mathbf{p}}{\hbar}$ and of a modified Bessel function of the second kind with the parameter $\frac{2i\mathbf{p}}{\hbar}$.

I faced a choice of physically acceptable solutions. I was aware of the fact that the spectrum of momentum was continuous. Hence the Wigner eigenfunctions of momentum should be generalized functions. I started from the definition of a functional action on test functions in presence of the $*-$ product. Formula (4.31) contains the trace density $t(q,p)$. In general it is extremely difficult to calculate this function (see [17], [32], [33]). Fortunately, in the analysed example the trace density $t(q,p) = 1$. Thus the $*-$ product was closed [34].

Finally I found the Wigner eigenfunctions of momentum. The spectrum of momentum is continuous and the Wigner eigenfunctions are given by formulas (4.34), (4.35) and (4.36). Notice that supports of Wigner eigenfunctions are: the closed half-plane $p \geq 0$ for positive eigenvalues $\mathbf{p} > 0$, the closed half-plane $p \leq 0$ for negative eigenvalues $\mathbf{p} < 0$ and the whole plane $\mathbb{R}^2$ for the eigenvalue $\mathbf{p} = 0$.

I investigated also the limit of the symplectic curvature 2− form (4.54), when the constant G^2 approaches 0. As it was expected, in this case every Wigner eigenfunction $W_{\mathbf{p}}$ tended to the respective Dirac delta $\delta(p - \mathbf{p})$.

I also analysed the case of a constant **positive** 2− form of a symplectic curvature. In such a phase space the spectrum of momentum is continuous but the Wigner eigenfunctions are defined by relations (4.61), (4.62) and (4.63) containing Bessel functions of the first kind. In the flat limit the Wigner eigenfunctions approach the well known flat solutions $\delta(p - \mathbf{p})$. As for the negative curvature case, the supports of the Wigner eigenfunctions are: the closed half-plane $p \geq 0$ for positive eigenvalues $\mathbf{p} > 0$, the closed half-plane $p \leq 0$ for negative eigenvalues $\mathbf{p} < 0$ and the plane $\mathbb{R}^2$ for $\mathbf{p} = 0$.

At the end I considered the problem of eigenvalues and Wigner eigenfunctions of position q. The eigenvalue equation for position separates in two equations: the equation (4.66) and the partial differential equation (4.67) of the infinite degree. I proved that the only solution of these pair of equations for the eigenvalue $\mathbf{q}$ is the generalised function $\delta(q - \mathbf{q})$. Hence the compatible symplectic connection determined by the coefficient $\Gamma_{111} = ap,\ a \in \mathbb{R}$ influences neither the eigenvalues nor the Wigner eigenfunctions of position.

3.6 Summary

For systems in the phase spaces $\mathbb{R}^{2n}$ deformation quantization is an autonomous formulation of quantum mechanics. This formulation is equivalent to the Hilbert space formulation. Hence development of deformation quantization is an alternative way towards the existing model.

However, for systems in more complicated phase spaces, deformation quantization seems to be the leading formalism. Thus it is extremely important to establish procedures and algorithms for solving concrete physical tasks in frames of deformation quantization. And this is the aim of my research.

Here is the list of my achievements mentioned in this review.

1. I proved the necessary and sufficient condition for an Abelian connection series to be finite. Applying this result I showed that in a curved 2–D symplectic manifold the Abelian connection series is always infinite. I proposed an example of a symplectic connection generating a finite Abelian connection.

2. I showed existence and I presented construction of compatible symplectic connections in cotangent bundles. The compatible symplectic connections seem to be the connections preferred by Physics, because they are related to symmetric linear connections in configuration spaces. I made a detailed analysis of the Fedosov construction for a compatible symplectic connection. I found three classes of terms appearing in the Abelian correction r. I proposed several properties of the $*-$ products induced by compatible symplectic connections.

3. I presented a geometric formulation of the Fedosov formalism. I proved that both: the Weyl algebra and the Weyl algebra bundle may be treated as infinite dimensional differential manifolds. I proposed construction of infinite dimensional matrices of a symplectic connection and of an Abelian connection in the Weyl bundle.

4. I prepared a computer program to perform symbolically the Fedosov $*-$ product in a Darboux chart. The user chooses the power of $\hbar$, the dimension of the phase space, the symplectic connection coefficients and the functions to be $*-$ multiplied.

5. At the example of a 1– D harmonic oscillator I showed, how to use the Fedosov method to calculate the Moyal product directly in an arbitrary chart. I applied this algorithm to solve the eigenvalue equation for the hamiltonian of the harmonic oscillator. I proposed several conditions to find physically acceptable solutions of an eigenvalue equation.

6. I considered the problem of spectrum and Wigner eigenfunctions of momentum and position in a 2–D phase space $\mathbb{R}^2$ with a constant curvature 2– form. I analysed the cases of a positive and negative curvature. I found the flat limit of these solutions receiving the results known from the flat symplectic space $\mathbb{R}^2$. I proposed the way of generalisation of the theory of generalised functions when a noncommutative closed $*-$ product is considered.

4 A brief review of my other scientific results

My research has been always related to quantum mechanics. I was especially interested in formulation of quantum theory of systems in nontrivial phase spaces. In my master thesis entitled 'Quantum mechanics in general relativity. Path integrals' I considered Feynman's approach to quantum physics.

When I took up a position at the Technical University of Lodz, I stopped working in path integrals and I moved for a moment to classical electrodynamics. I collaborated with prof. M. Przanowski and B. Rajca in the research on conservation laws in electrodynamics in a vacuum. We published paper [35] containing a construction of an infinite number of conservation laws generating according to the Noether theorem. We received the conservation law of energy of the electromagnetic field, the conservation law of momentum, so called zilch founded by Lipkin [36] and many other quantities which did not have a clear physical interpretation. Although article [35] had not been cited for several years, lately Y. Q. Tang and A. E. Cohen [37] quoted it.

I returned to quantum mechanical problems when I obtained prof. Plebański's notes about the Moyal formulation of quantum theory. At the first step I concentrated on inventing a simply and natural formula of ordering of operators called a generalized Weyl application [38]. Next I considered the inverse mapping known as a generalized Weyl correspondence [39]. Finally I found the outstanding paper [16] of Fedosov and since then I have been working in several aspects of the Fedosov model of quantization. My first paper about generalisation of Fedosov's construction [40] written with prof. M. Przanowski has been cited for 11 times (without self-citations, for the last time in 2010).

I would like to mention about my contribution to solving a problem of quantization on a cylinder $S^1 \times \mathbb{R}$. As it was shown in Refs. [41] and [42], the quantum phase space for a system with the classical phase space $S^1 \times \mathbb{R}$ is the space $S^1 \times \mathcal{Z}$. Hence the Stratonovich- Weyl quantizer should be defined in this latter space. Our result is in agreement with opinions presented by N. Mukunda [43] and M. V. Berry [44]. Paper [41] was widely commented by scientists working in quantization on the cylinder. This is one of the reasons why prof. M. A. del Olmo invited me to collaborate in preparation of article [42].

During my stay at the Valladolid University in Spain I worked in group theory. Together with prof. M. A. del Olmo and prof. J. Negro we published three papers [45]– [47]. We extended the $(2 + 1)-$ Poincare group and the $(2 + 1)-$ Galilei group by adding three electromagnetic components. As a result we constructed the $PM(2 + 1)$ Poincare– Maxwell group and the $GM(2 + 1)$ Galilei– Maxwell group. We tried to apply these new groups in the theory of anyons. Unfortunately, as in our considerations non canonical and non commutative coordinates appeared, it was difficult to find a physical interpretation of the results. Paper [47] has been cited for 9 times (without self-citations).

I also worked with prof. M. Przanowski in foundations of relativistic thermodynamics. Our inspiration were computer simulations of a relativistic gas. Recently we published paper [48] on this topic. We considered the problem of defining fundamental thermodynamic quantities in an inertial moving system of frames. We introduced two kinds of temperature: the empirical temperature present in the zeroth law of thermodynamics and the absolute temperature appearing in the second law of thermodynamics. We showed that although transformation rules for the heat and the absolute temperature depend on the method of measurement, the most natural seem to be the formulas proposed by Otto, Arzeliès and Möller. We also considered several statistical thermometers.

Now I am involved in two projects– how to eliminate unphysical solutions of an eigenvalue equation in frames of deformation quantization and how to introduce a time of arrival operator.

A necessary and sufficient condition for a real function to be a Wigner function of a pure state in the phase space $\mathbb{R}^2$ was published in [29]. A generalisation of this result in the space $\mathbb{R}^{2n}$ is natural.

On the latter problem I collaborate with prof. M. Przanowski and dr. M. Skulimowski. Our partial results were presented by prof. M. Przanowski at the XXX Workshop on Geometry and Physics in Bialowieza in 2011.

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